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A posteriori error estimates for subgrid viscosity stabilized approximations of convection–diffusion equations[☆]

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ABSTRACT

We derive a posteriori error estimates for subgrid viscosity stabilized finite element approximations of convection–diffusion equations in the high Péclet number regime. Two estimators are analyzed: an asymptotically robust one and a fully robust one with respect to the Péclet number. Numerical results on test cases with boundary layers or internal layers show that the asymptotically robust estimator can be used to construct adaptive meshes.

1. Introduction

The purpose of this work is to derive a posteriori error estimates for finite element approximations of steady reaction–convection–diffusion problems, with a special emphasis on the high Péclet number regime. It is well-known that approximating this type of equation by means of a standard Galerkin technique yields a solution that loses its H^1 stability when the cell Péclet numbers are large. This phenomenon is characterized by node to node oscillations of the numerical solution which are often amplified by nonlinearities. Many cures have been proposed in the literature, including in a conforming setting, streamline diffusion [1,2], subgrid viscosity [3,4], residual free bubbles [5], face penalty [6] and in a nonconforming or discontinuous setting [7–11]. In this work, we are interested in subgrid viscosity stabilization because it allows one to work in conforming settings and because it is easily extendable to unsteady problems and avoids the use of tunable parameters that depend on the cell Péclet numbers (as does face penalty stabilization).

The a posteriori error analysis of streamline-diffusion stabilized approximations to convection–diffusion equations in the high Péclet number regime is already well-understood owing to the pioneering work of Verfürth [12,13]; see also [14] for anisotropic meshes and [15] for face penalty stabilization. Two types of estimators can be derived, either asymptotically robust in the Péclet number [12] or fully robust [13]; see also [16–19] on robust a posteriori error estimators and [20] for hierarchical a posteriori error estimators. For the asymptotically robust estimator, local error bounds can be established using constants that involve a cut-off in terms of the Péclet number; for the fully robust estimator, the constants no longer

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depend on the Péclet number, but the norm in which the error is estimated is modified and cannot be localized in a straightforward manner, whereby only global lower error bounds are established.

In the present work we derive a posteriori error estimators for subgrid viscosity stabilized approximations to convection–diffusion equations. Following the techniques introduced by Verfürth [12,13], we analyze an asymptotically robust estimator and a fully robust estimator. The main difference with the streamline diffusion method analyzed in [12,13] is that subgrid viscosity stabilization yields a nonconsistent term that needs to be dealt with appropriately. The proofs below will pinpoint the treatment of this nonconsistency, while the treatment of the remaining terms will be skipped, since it is similar to that for the streamline diffusion method. This paper is organized as follows: Section 2 presents the setting under scrutiny. Section 3 deals with the analysis of the asymptotically robust estimator, while Section 4 deals with that of the fully robust estimator. Section 5 contains numerical results to illustrate how the asymptotically robust estimator can be used to construct adapted meshes in problems exhibiting boundary layers or internal layers. Conclusions are reached in Section 6.

2. The setting

Consider the following reaction–convection–diffusion problem with mixed Dirichlet–Neumann boundary conditions

$$\mathcal{L}u := -\varepsilon\Delta u + \beta\cdot\nabla u + \sigma u = f \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \Gamma_D, \tag{2}$$

$$\partial_n u = g \quad \text{on } \Gamma_N, \tag{3}$$

posed on a bounded polygonal or polyhedral domain $\Omega \subset \mathbb{R}^n, n = 2$ or 3, with boundary $\partial\Omega$ partitioned into $\partial\Omega = \Gamma_D \cup \Gamma_N$, Γ_D being of nonzero measure, and data $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$. We assume that $0 < \varepsilon \ll 1$, $\beta \in [W^{1,\infty}(\Omega)]^n$, $\sigma \in L^\infty(\Omega)$, $-\frac{1}{2}\nabla\cdot\beta+\sigma \geq \sigma_0 > 0$ on Ω, and $\Gamma_- := \{x \in \Gamma : \beta(x)\cdot n(x) < 0\} \subset \Gamma_D$. Letting $H^1_D(\Omega) = \left\{v \in H^1(\Omega);\ v|_{\Gamma_D} = 0\right\}$, the weak formulation of the above problem is to find $u \in H^1_D(\Omega)$ such that

$$a(u, v) = (f, v)_\Omega + (g, v)_{\Gamma_N}, \quad \forall v \in H^1_D(\Omega), \tag{4}$$

where

$$a(u, v) = (\varepsilon\nabla u, \nabla v)_\Omega + (\beta\cdot\nabla u + \sigma u, v)_\Omega. \tag{5}$$

Here and below, for any subset $R \subset \Omega$, $(\cdot,\cdot)_R$ denotes the usual $L^2(R)$-scalar product and $\|\cdot\|_R$ the associated norm. It is well-known that under the above assumptions, problem (4) is well-posed.

To approximate the unique solution of (4) using subgrid viscosity stabilized conforming finite elements, let $\mathcal{T}_h$, $h > 0$, denote a shape-regular mesh family of Ω into n-simplices. The meshes are assumed to be geometrically conforming, i.e., they cover Ω exactly and any two elements of the mesh are either disjoint or share a complete k-face with $0 \leq k \leq n-1$. Let

$$X^L_h = \{v^L_h \in H^1_D(\Omega), v^L_h|_T \in \mathbb{P}_1(T), \forall T \in \mathcal{T}_h\} \tag{6}$$

be the usual finite element space. It will play the role of the resolved scales space in the subgrid viscosity setting. Additionally, we define the subgrid scales space as

$$X^b_h = \mathrm{Span}_{T\in\mathcal{T}_h}\{\psi_T\}, \tag{7}$$

where for each mesh element $T \in \mathcal{T}_h$, ψ_T is the bubble function proportional to the product of barycentric coordinates on T; it vanishes on the boundary ∂T of T and takes the value 1 at the center of gravity of T. A generic element of X^b_h is denoted by v^b_h. Introduce the direct sum $X_h := X^L_h \oplus X^b_h$; a generic element $v_h \in X_h$ is decomposed as $v_h = v^L_h + v^b_h$ with $v^L_h \in X^L_h$ and $v^b_h \in X^b_h$. The discrete problem is to find $u_h \in X_h$ such that

$$a(u_h, v_h) + b_h(u^b_h, v^b_h) = (f, v_h)_\Omega + (g, v_h)_{\Gamma_N}, \quad \forall v_h \in X_h, \tag{8}$$

where

$$b_h(u^b_h, v^b_h) = \sum_{T\in\mathcal{T}_h} \varpi h_T(\nabla u^b_h, \nabla v^b_h)_T, \tag{9}$$

where ϖ is a user-dependent parameter that is independent of the mesh size and the problem data. In practice, for linear model problems, the parameter ϖ can be generally chosen in the range of unity. Moreover, problem (8) is well-posed; see, e.g., [3,4,21].

The following results will be used in the sequel. The proofs, which are straightforward, are omitted for brevity.

Lemma 2.1. *Any function $v_h \in X_h$ satisfies for all $T \in \mathcal{T}_h$,*

$$\|\nabla v^b_h\|_T \leq \|\nabla v_h\|_T. \tag{10}$$

Lemma 2.2. *Let u_h be the unique solution of (8). Then, for all $T \in \mathcal{T}_h$,*

$$\|\nabla u_h^b\|_T \le C\|\mathcal{L}u_h - f\|_T, \tag{11}$$

where the constant C only depends on the shape-regularity of the mesh family.

We are primarily concerned with the asymptotics of large Péclet numbers. We can assume, without loss of generality, that the problem at hand has been rescaled so that β is of order unity. We assume the same for σ and the parameter σ_0 since we are not interested in the asymptotics of strong reaction regimes. As a result, we will track the dependency of the constants in the a posteriori error estimates derived below with respect to the diffusion parameter ε and to the local mesh size. To alleviate the notation, we write $a \preceq b$ for the inequality $a \le cb$ where c is independent of ε and of any mesh size, but can depend on the shape-regularity of the mesh, the subgrid viscosity parameter ϖ, and the problem parameters β and σ. Similarly, $a \simeq b$ means $a \preceq b$ and $b \preceq a$.

3. Asymptotically robust a posteriori error estimate

The purpose of this section is to establish local lower and global upper bounds for the approximation error measured in the energy norm

$$\|\!|v|\!\|^2 = \varepsilon\,\|\nabla v\|_\Omega^2 + \|v\|_\Omega^2\,. \tag{12}$$

The following properties of the bilinear form a in terms of this energy norm will be useful: for all $v \in H_D^1(\Omega)$,

$$\|\!|v|\!\|^2 \preceq a(v, v), \tag{13}$$

and for all $v, w \in H_D^1(\Omega)$,

$$a(v, w) \preceq \|\!|v|\!\|(\|\!|w|\!\| + \varepsilon^{-\frac{1}{2}}\|w\|_\Omega). \tag{14}$$

Let $\mathcal{E}_h$ denote the set of all $(n-1)$-faces in $\mathcal{T}_h$. This set can be split into $\mathcal{E}_h = \mathcal{E}_{h,\Omega} \cup \mathcal{E}_{h,N} \cup \mathcal{E}_{h,D}$, where $\mathcal{E}_{h,\Omega}$, $\mathcal{E}_{h,N}$, and $\mathcal{E}_{h,D}$ refer to interior faces, Neumann boundary faces, and Dirichlet boundary faces, respectively. For all $E \in \mathcal{E}_{h,\Omega}$ and for all ϕ which is piecewise smooth, $[\phi]_E$ denotes the jump of ϕ across E (the sign of this quantity is irrelevant in the sequel). For all $S \in \mathcal{T}_h \cup \mathcal{E}_h$, let

$$\alpha_S = \min\{h_S\varepsilon^{-\frac{1}{2}}, 1\}, \tag{15}$$

where h_S denotes the diameter of S.

Denote by f_h, g_h, β_h, and σ_h the L^2-projection of the data f, g, β, and σ onto the space of piecewise constant functions on $\mathcal{T}_h$. Define the elementwise residual estimators as

$$\eta_T^2 = \alpha_T^2\|R_T\|_T^2 + \sum_{E\in\mathcal{E}_h;E\subset\partial T} \varepsilon^{-\frac{1}{2}}\alpha_E\|R_E\|_E^2, \tag{16}$$

where

$$R_T = f_h + \varepsilon\Delta u_h - \beta_h \cdot \nabla u_h - \sigma_h u_h, \tag{17}$$

and

$$R_E = \begin{cases} -[\varepsilon\partial_n u_h]_E & \text{if } E \in \mathcal{E}_{h,\Omega}, \\ g_h - \varepsilon\partial_n u_h & \text{if } E \in \mathcal{E}_{h,N}, \\ 0 & \text{if } E \in \mathcal{E}_{h,D}. \end{cases} \tag{18}$$

Define also the oscillation term for $T \in \mathcal{T}_h$ by

$$D_T = (f - f_h) + (\beta - \beta_h) \cdot \nabla u_h + (\sigma - \sigma_h)u_h, \tag{19}$$

and the Neumann oscillation term for $E \in \mathcal{E}_{h,N}$ by

$$D_E = g - g_h. \tag{20}$$

Finally, define the elementwise data oscillation estimator as

$$\Theta_T^2 = \alpha_T^2\|D_T\|_T^2 + \sum_{E\in\mathcal{E}_{h,N};E\subset\partial T} \varepsilon^{-\frac{1}{2}}\alpha_E\|D_E\|_E^2. \tag{21}$$

Theorem 3.1. *Let u and u_h be the unique solutions of (4) and (8), respectively. Then,*

$$|||u-u_h|||^2 \preceq \sum_{T\in\mathcal{T}_h}[\eta_T^2+\Theta_T^2]. \tag{22}$$

Proof. Owing to (13),

$$\begin{aligned}|||u-u_h||| &\preceq \sup_{v\in H_D^1(\Omega)\setminus\{0\}}\frac{a(u-u_h,v)}{|||v|||}\\ &\le \sup_{v\in H_D^1(\Omega)\setminus\{0\}}\frac{a(u-u_h,v-I_hv)}{|||v|||}+\sup_{v\in H_D^1(\Omega)\setminus\{0\}}\frac{a(u-u_h,I_hv)}{|||v|||},\end{aligned}$$

where $I_h : L^2(\Omega)\to X_h$ is the quasi-interpolation operator of Clément introduced by Verfürth in [12]. The first term in the above right-hand side is treated as usual and yields

$$a(u-u_h,v-I_hv)\preceq\left\{\sum_{T\in\mathcal{T}_h}[\eta_T^2+\Theta_T^2]\right\}^{\frac12}|||v|||.$$

The new contribution is the second term; it results from the nonconsistency of the subgrid viscosity stabilization. Let $w_h := I_hv$ and observe that

$$a(u-u_h,w_h)=-\sum_{T\in\mathcal{T}_h}\varpi h_T(\nabla u_h^b,\nabla w_h^b)_T\le\sum_{T\in\mathcal{T}_h}\varpi h_T\|\nabla u_h^b\|_T\|\nabla w_h^b\|_T.$$

Using Lemma 2.1 and a standard scaling argument yield for all $T\in\mathcal{T}_h$,

$$\|\nabla w_h^b\|_T\le\|\nabla w_h\|_T\preceq\alpha_T h_T^{-1}|||w_h|||_T,$$

where $|||w_h|||_T^2=\varepsilon\|\nabla w_h\|_T^2+\|w_h\|_T^2$. Using the stability of I_h in the energy norm [12] and Lemma 2.2 leads to

$$\begin{aligned}a(u-u_h,I_hv) &\preceq\left\{\sum_{T\in\mathcal{T}_h}\alpha_T^2\,\|f-\mathcal{L}u_h\|_T^2\right\}^{\frac12}|||v|||\\ &\preceq\left\{\sum_{T\in\mathcal{T}_h}\alpha_T^2\,\|R_T\|_T^2+\sum_{T\in\mathcal{T}_h}\alpha_T^2\,\|D_T\|_T^2\right\}^{\frac12}|||v|||,\end{aligned}$$

whence the conclusion readily follows. □

To conclude this section, we bound locally the residual estimator η_T in terms of the approximation error and the data oscillation estimator. For $T\in\mathcal{T}_h$, let

$$\omega_T=\bigcup_{\emptyset\ne T\cap T'\in\mathcal{E}_{h,\Omega}}T', \tag{23}$$

and set $\Theta_{\omega_T}^2=\sum_{T'\in\omega_T}\Theta_{T'}^2$. The same notation is used for indexing the energy norm. Using the technique of bubble functions with cut-off depending on the quantity α_S defined by (15), the following result can be readily proven. The proof is skipped since the use of subgrid viscosity stabilization does not introduce any substantial modification with respect to [12].

Theorem 3.2. *For all $T\in\mathcal{T}_h$, there holds*

$$\eta_T\preceq(1+\varepsilon^{-\frac12}\alpha_T)|||u-u_h|||_{\omega_T}+\Theta_{\omega_T}. \tag{24}$$

4. Robust a posteriori error estimate

The purpose of this section is to derive a robust a posteriori error estimate in the context of subgrid viscosity stabilized finite element approximations. To this purpose, we follow the approach proposed by Verfürth [13] and aim at measuring the approximation error in the new norm

$$|||v|||_*=|||v|||+\|\beta\cdot\nabla v\|_{-1,\varepsilon}, \tag{25}$$

where for all $\varphi\in H^{-1}(\Omega)$,

$$\|\varphi\|_{-1,\varepsilon}=\sup_{v\in H_D^1(\Omega)\setminus\{0\}}\frac{\langle\varphi,v\rangle}{|||v|||}, \tag{26}$$

and $\langle\cdot,\cdot\rangle$ denotes the corresponding duality pairing. The following stability and continuity properties of the bilinear form a are readily verified: For all $v \in H^1_D(\Omega)$,

$$\|\!|v|\!\|_* \preceq \sup_{w\in H^1_D(\Omega)\setminus\{0\}} \frac{a(v,w)}{\|\!|w|\!\|}, \tag{27}$$

and for all $v, w \in H^1_D(\Omega)$,

$$a(v,w) \le \|\!|v|\!\|_* \|\!|w|\!\|. \tag{28}$$

Theorem 4.1. *Let u and u_h be the unique solutions of (4) and (8), respectively. Then,*

$$\|\!|u-u_h|\!\|_*^2 \preceq \sum_{T\in\mathcal{T}_h} [\eta_T^2 + \Theta_T^2]. \tag{29}$$

Proof. We define the residual $R(u_h) \in H^{-1}(\Omega)$ of the discrete solution by

$$\langle R(u_h), v\rangle = a(u-u_h, v), \quad \forall v \in H^1_D.$$

Owing to (27),

$$\|\!|u-u_h|\!\|_* \preceq \|R(u_h)\|_{-1,\varepsilon} \le \sup_{v\in H^1_D(\Omega)\setminus\{0\}} \frac{a(u-u_h, v-I_hv)}{\|\!|v|\!\|} + \sup_{v\in H^1_D(\Omega)\setminus\{0\}} \frac{a(u-u_h, I_hv)}{\|\!|v|\!\|}.$$

To conclude, observe that both terms have been estimated in the proof of Theorem 3.1. □

A global lower bound for the approximation error measured in the $\|\!|\cdot|\!\|_*$-norm augmented by the data oscillation term can be proven, as in [13]. The proof is skipped since the nonconsistency of the approximation method poses no difficulties.

Theorem 4.2. *The following holds:*

$$\sum_{T\in\mathcal{T}_h} \eta_T^2 \preceq \|\!|u-u_h|\!\|_*^2 + \sum_{T\in\mathcal{T}_h} \Theta_T^2. \tag{30}$$

5. Numerical results

The purpose of this section is to illustrate by numerical examples that the asymptotically robust a posteriori estimator derived in Section 3 can be used to construct adaptive meshes. In the test cases, we take $\Omega = [0,1]^2$, $\beta = (1,1)^t$, $\sigma = 1$, and we impose homogeneous Dirichlet boundary conditions on the whole boundary. The subgrid viscosity parameter is set to $\varpi = 1$. The meshes are refined using the following criterion: For each $T \in \mathcal{T}_h$, we compute the estimator η_T of the error in T. Letting $\eta = \max_{T\in\mathcal{T}_h} \eta_T$, an element T is subdivided if $\eta_T \ge \tau\eta$, where $\tau = 0.25$. The initial mesh is always quasi-uniform.

5.1. A test case with boundary layers

In this first test case, we take the right-hand side f such that the exact solution is

$$u(x,y) = xy(1-\mathrm{e}^{(x-1)/\gamma})(1-\mathrm{e}^{(y-1)/\gamma}), \tag{31}$$

with boundary layer parameter $\gamma = 0.05$. The diffusion parameter is set to $\varepsilon = 10^{-6}$. Fig. 1 presents adaptive meshes. The refinement occurs at both boundary layers located near the lines $\{x = 1\}$ and $\{y = 1\}$ as expected. Fig. 2 presents the error measured in the energy norm and the asymptotically robust estimator as a function of degrees of freedom in logarithmic scales. Both the error and the estimator are normalized by the value they take on the initial mesh. The scaling factor between both quantities is as expected of the order of $\varepsilon^{-1/2}$. We observe that the convergence rate of the error and the estimator is quite close for the adaptive algorithm.

5.2. A test case with internal layer

In this second test case, we take the right-hand side f such that the exact solution is

$$u(x,y) = xy(1-x)(1-y)(1+\tanh(\gamma^{-1}(x^2+y^2-0.25))), \tag{32}$$

with interior layer parameter $\gamma = 0.01$. The diffusion parameter is set to $\varepsilon = 10^{-6}$. Fig. 3 presents adaptive meshes. The refinement occurs at the internal layer as expected. Fig. 4 displays the convergence history of the error measured in the energy norm and of the estimator, using the same setting as in Fig. 2. Similar conclusions to those for the previous test case can be drawn.

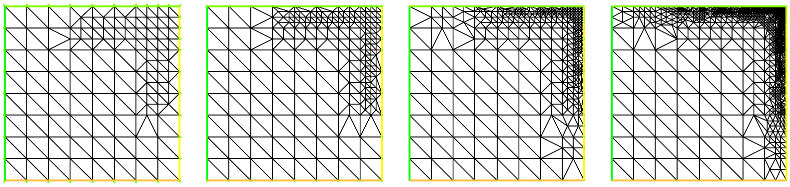

Fig. 1. Adaptive meshes for the test case with boundary layers.

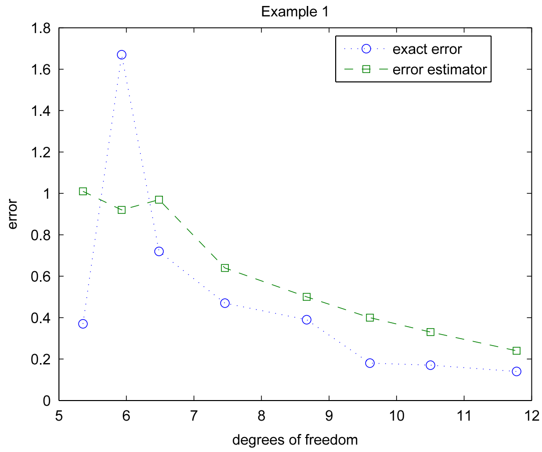

Fig. 2. Error and estimator convergence.

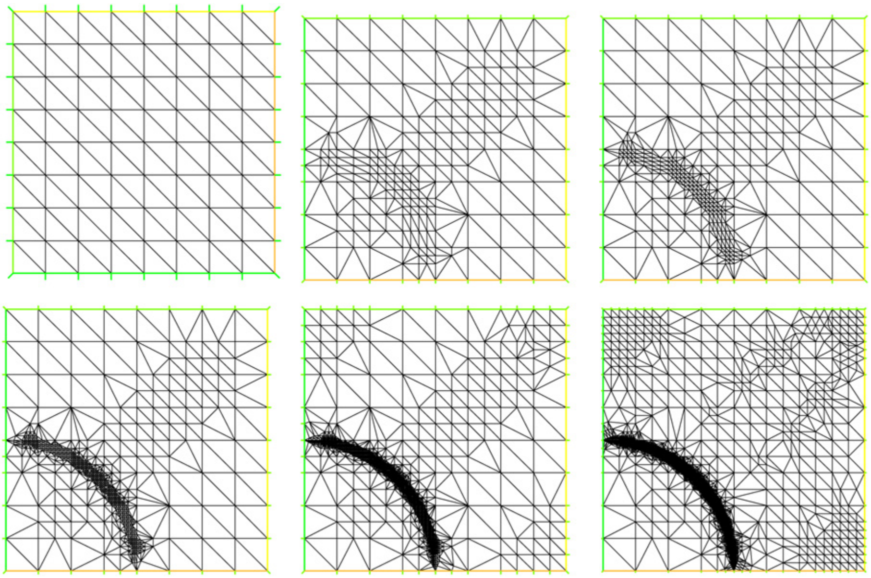

Fig. 3. Adaptive meshes for the test case with internal layer.

6. Conclusion

In this work, we have analyzed two a posteriori error estimators for subgrid viscosity stabilized finite element approximations to convection–diffusion equations in the high Péclet number regime. In the spirit of previous work by Verfürth [12,13], the first estimator is asymptotically robust with respect to the Péclet number, while the second one is fully robust. Numerical examples on test cases with boundary layers and internal layers have shown that the present estimators can be used to construct adaptive meshes.

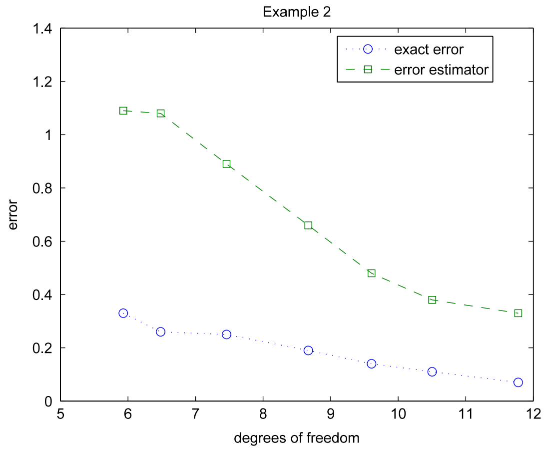

Fig. 4. Error and estimator convergence.

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